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Cardinality bounds for subdirectly irreducible algebras

Keith A. Kearnes *

Department of Mathematical Sciences, University of Arkansas, Fayetteville, AR 72701, USA

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Abstract

In this paper we show that, if $\mathscr{V}$ is a residually small variety generated by an algebra with $n < \omega$ elements, and $\mathbf{A}$ is a subdirectly irreducible algebra in $\mathscr{V}$ with restricted type labeling, then $|A| \leq n^{n^{n^{n+2}}}$.

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1. Introduction

The two fundamental representation theorems for varieties of algebras are the HSP Theorem and the Subdirect Representation Theorem, both due to Birkhoff. The HSP Theorem states that the variety generated by a class K of similar algebras is the smallest class of algebras containing K which is closed under the formation of homomorphic images, subalgebras and products. The proof of the HSP Theorem shows in fact that

$$\mathsf{V}(K) = \mathsf{HSP}(K).$$

That is, if one closes under products, then subalgebras and finally under the formation of homomorphic images, then one obtains a class of algebra closed under all three constructions. This shows that an arbitrary member of $\mathsf{V}(K)$ may be represented as $\mathbf{B}/\theta$ where θ is a congruence on $\mathbf{B}$ and $\mathbf{B}$ is a subalgebra of $\prod_{i\in I}\mathbf{A}_i$, $\mathbf{A}_i \in K$. Since $\mathbf{B} \leq \prod_{i\in I}\mathbf{A}_i$, $\mathbf{B}$ is simply an algebra of vectors where, for each i, the values in the i^{th} coordinate of a vector are from some fixed $\mathbf{A}_i \in K$. Therefore any member of $\mathsf{V}(K)$ may be considered to be an algebra of equivalence classes of vectors with coordinate algebras from K. Now, while it may be fairly easy to calculate coordinatewise with

* E-mail: Kearnes@comp.uark.edu. Research supported by a fellowship from the Alexander von Humboldt Stiftung.

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vectors, it is usually quite difficult to calculate with equivalence classes of vectors. This difficulty is addressed by the Subdirect Representation Theorem.

The Subdirect Representation Theorem states that any member of a variety $\mathscr{V}$ is isomorphic to a subdirect product of subdirectly irreducible members of $\mathscr{V}$. This implies that $\mathscr{V} = \mathsf{SP}(Si)$ where Si is the class of subdirectly irreducible algebras in $\mathscr{V}$. Where the HSP Theorem represents the members of $\mathscr{V} = \mathsf{V}(K)$ as algebras of equivalence classes of vectors, the Subdirect Representation Theorem represents these algebras as algebras of vectors. The latter representation is easier to work with, but it requires knowing the class of subdirectly irreducible members of $\mathscr{V}$. This leads naturally to the following problem: Given a class K of similar algebras, describe the subdirectly irreducible members of $\mathsf{V}(K)$.

In many cases, it is a hopeless task to describe the subdirectly irreducible members of $\mathsf{V}(K)$, even when K is well-understood. The case when $K = \{\mathbf{A}\}$ consists of a single finite algebra has received the most attention. Here the approach has been to prove general theorems which either (i) show that $\mathsf{V}(\mathbf{A})$ has a proper class of subdirectly irreducibles or (ii) produce a finite cardinality bound on the size of the subdirectly irreducible algebras in $\mathsf{V}(\mathbf{A})$. Some theorems have been found which have a fairly general scope, but this type of approach leads one to wonder if there are finite algebras $\mathbf{A}$ which fit into neither category. Indeed, versions of the following conjecture concerning the distribution of subdirectly irreducible algebras remained open for more than 20 years. (To explain the wording, a variety is *residually large* if it has a proper class of isomorphism types of subdirectly irreducible algebras. Otherwise it is *residually small.*)

The RS Conjecture. *If* $\mathbf{A}$ *is a finite algebra and* $\mathsf{V}(\mathbf{A})$ *is residually small, then there is a finite bound on the size of its subdirectly irreducible members.*

The conjecture states that if $\mathbf{A}$ is finite and $\mathsf{V}(\mathbf{A})$ has some bound on the cardinality of its subdirectly irreducible members, then it has a finite bound. This is sometimes expressed as, 'If $\mathbf{A}$ is finite and $\mathsf{V}(\mathbf{A})$ is residually small, then $\mathsf{V}(\mathbf{A})$ is residually $<< \omega$.'

Attempts to prove the RS Conjecture led to a vigorous investigation of the combinatorics of finite algebras which continues today. We are referring to what is called *tame congruence theory* and [3] is the handbook of the theory. Tame congruence theory associates with each covering pair of congruences a number from one to five. This number explains the local behavior of polynomial operations with respect to the chosen congruences. The number is called the *type* of the covering. The set of all numbers associated with a finite algebra $\mathbf{A}$ is called the *type-set* of $\mathbf{A}$ and it is written typ$\{\mathbf{A}\}$. We write typ$\{\mathsf{V}(\mathbf{A})\}$ to denote the set of all type labels associated with finite members of $\mathsf{V}(\mathbf{A})$. In all cases the type-set of an algebra or variety is a subset of $\{\mathbf{1},\mathbf{2},\mathbf{3},\mathbf{4},\mathbf{5}\}$.

Although there are many papers classifying the residually small subvarieties of certain well-known varieties, we mention only a few of the important results which led up to this paper. Not all of these results were proved with tame congruence theory, but we state the results in the language of tame congruence theory so that a comparison can be made.

(1967) Jónsson's Lemma (see [4]) implies that the RS conjecture holds if typ$\{\mathsf{V}(\mathbf{A})\} \subseteq \{\mathbf{3},\mathbf{4}\}$ and all minimal sets have empty tail.

(1981) The paper [1] by Freese and McKenzie proves, among other things, that the RS conjecture holds if typ$\{\mathsf{V}(\mathbf{A})\} \subseteq \{\mathbf{2},\mathbf{3},\mathbf{4}\}$ and all minimal sets have empty tail.

(1983) Hobby and McKenzie prove that the RS conjecture holds if typ$\{\mathsf{V}(\mathbf{A})\} \subseteq \{\mathbf{2},\mathbf{3},\mathbf{4}\}$.

(1986) McKenzie proves that in a finitely generated residually small variety for which typ$\{\mathscr{V}\} \subseteq \{\mathbf{1},\mathbf{2},\mathbf{3},\mathbf{4}\}$ there is a finite cardinality bound which holds for all subdirectly irreducible algebras whose monolith is of type **2**, **3** or **4**.

(1991) The author proves that in any finitely generated residually small variety there is a finite cardinality bound which holds for all subdirectly irreducible algebras which omit type **5** and whose monolith is of type **2**, **3** or **4**. This bound depends only on the size of the generating algebra.

This paper contains a proof of the last result. On the surface the statement of this result seems to be a small improvement over the preceding two results, but it is the first result obtained in this area which requires no global restriction on the variety, i.e., no type restrictions on the variety are assumed.

The last two results on this list were not published at the time of their discovery. The hope was that these ideas would form a part of an eventual proof of the RS conjecture. However, in 1993, while attempting to extend the ideas from his 1986 proof, McKenzie discovered a counterexample to the RS conjecture. Indeed, he went on to produce a sequence of even more startling counterexamples until he announced that he could interpret the halting problem into the problem of determining if $\mathsf{V}(\mathbf{A})$ is residually $<< \omega$ for finite $\mathbf{A}$ (see [7]). Thus, the class of finite algebras which generate varieties which are residually $<< \omega$ is recursively inseparable from the class of algebras which generate residually countable varieties which are not residually $<< \omega$. Since then, McKenzie and Willard have shown that the class of finite algebras which generate varieties which are residually $<< \omega$ is recursively inseparable from the class of algebras which generate residually large varieties. McKenzie has also shown that, for a finite algebra $\mathbf{A}$, if $\mathsf{V}(\mathbf{A})$ has a cardinality bound on its subdirectly irreducible members, then that bound can be anything permitted by the early model-theoretic restrictions discovered by Taylor [10] and McKenzie and Shelah [9]. About the only conjecture in this area that McKenzie did not solve negatively is the following one (which remains open).

Conjecture. *If* $\mathbf{A}$ *is a finite algebra with finitely many basic operations and every subdirectly irreducible algebra in* $\mathsf{V}(\mathbf{A})$ *is finite, then* $\mathsf{V}(\mathbf{A})$ *is residually* $<< \omega$.

All of McKenzie's new examples involve a heavy dependence on the pathology of type **5** quotients in finite algebras. It seems the appropriate time to publish our positive results on residual smallness, since it is now clear that good positive results cannot be obtained for subdirectly irreducible algebras with type **5** quotients. Our theorem does give good positive results for almost any subdirectly irreducible algebra which omits type **5**. (In our main theorem we permit all types other than **5**, except we do not allow the monolith to have a type **1** label.) It is still an intriguing question as to whether the RS conjecture holds for varieties with no type **5** quotients.

Throughout this paper we make free use of tame congruence theory. The reader is directed to [3] for the terminology and results of the theory.

2. Large subdirectly irreducible algebras

In this paper we are investigating finite algebras $\mathbf{A}$ for which there is a cardinality bound on the size of subdirectly irreducible algebras in $\mathsf{V}(\mathbf{A}) = \mathsf{HSP}(\mathbf{A})$. We shall find it more convenient to calculate in $\mathsf{SP}(\mathbf{A})$ rather than $\mathsf{HSP}(\mathbf{A})$. We need to be able to recognize from the members of $\mathsf{SP}(\mathbf{A})$ whether or not there will be large subdirectly irreducible algebras in $\mathsf{HSP}(\mathbf{A})$. Thus, rather than work with large subdirectly irreducible algebras directly, we shall work with algebras which have large subdirectly irreducible homomorphic images. The next lemma, which is a basic tool, gives a necessary and sufficient condition for an algebra to have a large subdirectly irreducible homomorphic image.

Lemma 2.1. *An algebra* $\mathbf{B}$ *has a subdirectly irreducible homomorphic image of cardinality* $\geq \kappa$ *if and only if there is a 4-tuple* (a,b,X,γ) *satisfying the following conditions*:

(i) $a,b \in B$, $X \subseteq B$,

(ii) $\gamma \in \mathrm{Con}\,\mathbf{B}$ *and* $(a,b) \notin \gamma$,

(iii) *for every* $\psi \in \mathrm{Con}\,\mathbf{B}$ *with* $\psi \geq \gamma$ *the following implication holds:*

$$|X/(\psi|_X)| < \kappa \Rightarrow (a,b) \in \psi .$$

Proof. If $\mathbf{B}$ has a homomorphism onto a subdirectly irreducible of cardinality $\geq \kappa$, then choose γ to be the kernel of the homomorphism. Necessarily γ is completely meet-irreducible. Let γ^* denote the upper cover of γ. Choose $a,b \in B$ so that $(a,b) \in \gamma^* - \gamma$ and let X be any transversal for γ. Note that $|X| = |A/\gamma| \geq \kappa$. Note also that the only $\psi \geq \gamma$ for which $(a,b) \notin \psi$ is $\psi = \gamma$ and for this value of ψ we have $\psi|_X = 0_X$, since X is a transversal for $\gamma = \psi$. Hence, for any $\psi \geq \gamma$ we have $|X/(\psi|_X)| < \kappa$ implies $(a,b) \in \psi$.

For the other direction, assume that there exists a 4-tuple (a,b,X,γ) satisfying the prescribed conditions. Choose any $\psi \geq \gamma$ maximal for the property that $(a,b) \notin \psi$. The maximality of ψ implies that $\mathbf{B}/\psi$ is subdirectly irreducible while condition (iii) of the lemma guarantees that

$$|B/\psi| \geq |X/(\psi|_X)| \geq \kappa.$$

Hence, $\mathbf{B}$ has a subdirectly irreducible homomorphic image of cardinality $\geq \kappa$. $\square$

In the rest of this paper, whenever we have to prove that a variety of the form $\mathsf{V}(\mathbf{A})$ has a proper class of subdirectly irreducible algebras, we shall find it sufficient to produce for each κ an algebra $\mathbf{B}_\kappa \in \mathsf{SP}(\mathbf{A})$ which has a 4-tuple (a, b, X, γ) satisfying conditions (i) – (iii) of Lemma 2.1.

3. Generalizing Jónsson's Lemma

Our goal in this section is to extend Jónsson's Lemma to arbitrary finitely generated, residually small varieties. The classical version of Jónsson's Lemma for finitely generated, congruence distributive varieties is:

Lemma 3.1 (Jónsson's Lemma [4]). *Let K be a finite set of finite algebras such that $\mathsf{V}(K)$ is congruence distributive. If $\mathbf{A} \in \mathsf{V}(K)$ is subdirectly irreducible, then $\mathbf{A} \in \mathsf{HS}(K)$.*

A generalization of this lemma to congruence modular varieties appears in [2]. A version of that result for finitely generated, congruence modular varieties is the following. (In this statement $(0 : \mu)$ denotes the largest congruence θ such that $[\theta, \mu] = 0$.)

Lemma 3.2. *Let K be a finite set of finite algebras such that $\mathsf{V}(K)$ is congruence modular. If $\mathbf{A} \in \mathsf{V}(K)$ is a finite subdirectly irreducible with monolith μ, then $\mathbf{A}/(0 : \mu) \in \mathsf{HS}(K)$.*

The Jónsson's-type lemma that we shall prove in this section is

Lemma 3.3. *Let K be a finite set of finite algebras. Assume that $\mathsf{V}(K)$ is residually small. If $\mathbf{A} \in \mathsf{V}(K)$ is a finite subdirectly irreducible with monolith μ and*

(i) $\mathbf{5} \notin \operatorname{typ}\{\mathbf{A}\}$,

(ii) $\operatorname{typ}(0, \mu) \neq \mathbf{1}$;

then $\mathbf{A}/(0 : \mu) \in \mathsf{HS}(K)$.

The following result is a first step to proving our Jónsson's-type lemma.

Lemma 3.4. *Let $\mathbf{A}$ be a finite algebra which has congruences $\delta \prec \theta$ and η_i, $i < n$, such that $\bigwedge_{i<n} \eta_i \leq \delta$. If $\operatorname{typ}(\delta, \theta) \in \{\mathbf{2}, \mathbf{3}, \mathbf{4}\}$, then $C(\theta, \eta_i; \delta)$ holds for some i.*

Proof. What we actually prove is that if $\operatorname{typ}(\delta, \theta) \in \{\mathbf{2}, \mathbf{3}, \mathbf{4}\}$ and η is any congruence on $\mathbf{A}$ where $N^2 \nsubseteq (\eta \cup \delta)$ for some $\langle \delta, \theta \rangle$-trace N, then $C(\theta, \eta; \delta)$ holds. This will suffice to prove the lemma as we now explain. If N is a $\langle \delta, \theta \rangle$-trace, $(u, v) \in N^2 - \delta$ and $\bigwedge_{i<n} \eta_i \leq \delta$; then $(u, v) \notin \eta_i$ must hold for some i. Thus, $(u, v) \in N^2 - (\eta_i \cup \delta)$ for some i. Proving that $N^2 \nsubseteq (\eta \cup \delta)$ implies $C(\theta, \eta; \delta)$ will establish the lemma.

Assume that $N^2 \nsubseteq (\eta \cup \delta)$ for some N and some η. Choose $U \in M_{\mathbf{A}}(\delta, \theta)$ containing N and choose $(u, v) \in N^2 - (\eta \cup \delta)$. Assume that $C(\theta, \eta; \delta)$ fails. Then, since $\theta =$

$\mathrm{Cg}(u,v) \vee \delta$, $C(\{(u,v)\},\eta;\delta)$ fails too. Therefore, there is a polynomial $p(x,\bar{y}) \in \mathrm{Pol}_{m+1}\mathbf{A}$ and elements $(a_j,b_j) \in \eta$ such that

$$p(u,\bar{a})\ \delta\ p(u,\bar{b}),$$

while

$$g = p(v,\bar{a})\ \theta - \delta\ p(v,\bar{b}) = h$$

(or else the same statement holds with u and v switched). All four of the elements in these two equations belong to the same θ-class. Choose $f \in \mathrm{Pol}_1\mathbf{A}$ such that $f(A) \subseteq U$ and $(f(g),f(h)) \notin \delta$. Composing f with p we may assume $p(A^{m+1}) \subseteq U$ and therefore that the four elements of the previous displayed equations all belong to the body of U. If $\mathrm{typ}(\delta,\theta) \in \{\mathbf{3},\mathbf{4}\}$, then the four elements in the last two displayed equations are among the two distinct elements of the body of U and these elements are u and v. Hence, $\{p(v,\bar{a}),p(v,\bar{b})\} = \{u,v\}$. But $(p(v,\bar{a}),p(v,\bar{b})) \in \eta$ while $(u,v) \notin \eta$. This contradiction shows that $\mathrm{typ}(\delta,\theta) \notin \{\mathbf{3},\mathbf{4}\}$ if $C(\theta,\eta;\delta)$ fails. We are forced to conclude that $\mathrm{typ}(\delta,\theta) = \mathbf{2}$. Let $d(x,y,z)$ be a pseudo-Mal'cev polynomial of U. We may assume that $d(A^3) \subseteq U$. Let $q(x,\bar{y}) = d(p(x,\bar{y}),p(x,\bar{b}),p(v,\bar{b}))$. Then

$$q(u,\bar{b}) = p(v,\bar{b}) = q(v,\bar{b})$$

while

$$q(u,\bar{a})\ \delta\ p(v,\bar{b})\ \theta - \delta\ p(v,\bar{a}) = q(v,\bar{a}).$$

All elements in these equations belong to the body of U. Now define a polynomial $r(x,\bar{y}) = d(q(x,\bar{y}),q(u,\bar{y}),q(u,\bar{b}))$. We have

$$r(u,\bar{b}) = q(u,\bar{b}) = r(v,\bar{b})$$

while

$$r(u,\bar{a}) = q(u,\bar{b})\ \delta\ q(u,\bar{a})\ \theta - \delta\ q(v,\bar{a})\ \delta\ r(v,\bar{a}).$$

Since $r(\theta|_U,\bar{a}) \nsubseteq \delta|_U$, $r(x,\bar{a})$ is a permutation of U. Let $r_{\bar{a}}^{-1}$ be a polynomial inverse to $r(x,\bar{a})$ on U. Then we have

$$u = r_{\bar{a}}^{-1}r(u,\bar{a}) = r_{\bar{a}}^{-1}r(u,\bar{b}) = r_{\bar{a}}^{-1}r(v,\bar{b})$$

while $r_{\bar{a}}^{-1}r(v,\bar{a}) = v$. In particular, $(u,v) = (r_{\bar{a}}^{-1}r(v,\bar{b}),r_{\bar{a}}^{-1}r(v,\bar{a})) \in \eta$. Again we face the same contradiction: u and v were chosen so that $(u,v) \notin \eta$. This contradiction proves the lemma. □

It will be worth our while to show now how Lemma 3.4 can be used to prove the finitely generated version of the classical Jónsson's Lemma as well as its generalization to congruence modular varieties. This will suggest what further work is necessary in order to establish our Jónsson's-type lemma.

Proof of Lemmas 3.1 and 3.2. Let $\mathbf{A}$ be a finite subdirectly irreducible in $\mathsf{V}(K)$. Then $\mathbf{A}$ may be represented as $\mathbf{B}/\delta$ where $\mathbf{B}$ is a subalgebra of some finite product $\prod_{j<n}\mathbf{C}_j$, $\mathbf{C}_j \in K$, and δ is a congruence on $\mathbf{B}$. Let η_i be the congruence on $\mathbf{B}$ which is the restriction to $\mathbf{B}$ of the kernel of the ith projection $\pi_i : \prod_{j<n}\mathbf{C}_j \to \mathbf{C}_i$. Since $\mathbf{B}$ is embedded in $\prod_{j<n}\mathbf{C}_j$ we have $\bigwedge_{i<n}\eta_i = 0 \le \delta$. Since $\mathbf{B}/\delta \cong \mathbf{A}$ is subdirectly irreducible, δ has a unique upper cover in $\mathbf{Con\ B}$ which we label θ. In Lemmas 3.1 and 3.2 we are in a congruence modular variety, so by Theorem 8.5 of [3] we must have $\mathrm{typ}(\delta,\theta) \in \{\mathbf{2},\mathbf{3},\mathbf{4}\}$. Now we may use Lemma 3.4 to conclude that for some i we have $C(\theta,\eta_i;\delta)$.

In a congruence modular variety the centralizer relation is symmetric in its first two variables. In fact, in a congruence modular variety we have

$$\begin{aligned} C(\theta,\eta_i;\delta) &\Leftrightarrow [\theta,\eta_i] \le \delta \\ &\Leftrightarrow [\eta_i,\theta] \le \delta \\ &\Leftrightarrow C(\eta_i,\theta;\delta). \end{aligned}$$

These bi-implications are proved in [2, Ch. 4]. Thus, from $C(\theta,\eta_i;\delta)$ we deduce that $\eta_i \le (\delta : \theta)$. From the Second Isomorphism Theorem we have

$$\mathbf{B}/(\delta : \theta) \in \mathsf{H}(\mathbf{B}/\eta_i) \subseteq \mathsf{HS}(\mathbf{C}_i) \subseteq \mathsf{HS}(K).$$

If μ is the monolith of $\mathbf{A}$, then (since θ/δ is the monolith of $\mathbf{B}/\delta \cong \mathbf{A}$) we have $\mathbf{A}/(0 : \mu) \cong \mathbf{B}/(\delta : \theta)$. Hence, $\mathbf{A}/(0 : \mu) \in \mathsf{HS}(K)$ which proves Lemma 3.2.

In Jónsson's Lemma, we even have that $\mathsf{V}(K)$ is congruence distributive. As is shown in Exercise 1 of [2, Ch. 8], the commutator equal the intersection in this case, so $(0 : \mu) = 0$ in $\mathbf{A}$. Thus,

$$\mathbf{A} \cong \mathbf{A}/(0 : \mu) \in \mathsf{HS}(K).$$

This proves that every finite subdirectly irreducible in $\mathsf{V}(K)$ is contained in $\mathsf{HS}(K)$. But, this imposes a finite cardinality bound on the finitely generated subdirectly irreducibles in $\mathsf{V}(K)$. By Lemma 10.2 of [2], $\mathsf{V}(K)$ has no infinite subdirectly irreducibles. Thus, every subdirectly irreducible member of $\mathsf{V}(K)$ is contained in $\mathsf{HS}(K)$. This proves Lemma 3.1. □

Looking over the proof of Lemma 3.2 we find that there are exactly two places where we used the assumption that $\mathsf{V}(K)$ is congruence modular. We first used it to deduce that $\mathrm{typ}(\delta,\theta) \in \{\mathbf{2},\mathbf{3},\mathbf{4}\}$. We later used it to deduce from $C(\theta,\eta_i;\delta)$ that $C(\eta_i,\theta;\delta)$ holds. This indicates that most of this proof works without any modularity assumption if

(i) we restrict our attention only to subdirectly irreducible algebras $\mathbf{A}$ where $\mathrm{typ}(0,\mu) \in \{\mathbf{2},\mathbf{3},\mathbf{4}\}$ (since $\mathrm{typ}(0,\mu) = \mathrm{typ}(\delta,\theta)$ in the above proof), and

(ii) we find some other way to deduce from $C(\theta,\eta_i;\delta)$ that $C(\eta_i,\theta;\delta)$ holds.

This is what we intend to do. We shall outline our strategy for the proof of Lemma 3.3 in the next few paragraphs using the notation of the previous proof.

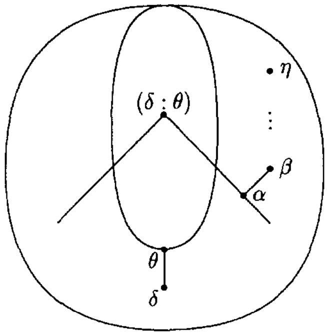

Fig. 1. Con B.

The precise relationship between $C(\theta,\eta;\delta)$ and $C(\eta,\theta;\delta)$ when η,θ and δ are congruences on a finite algebra and $\delta \prec \theta$ is explained in [6]. The following result is proved there.

Theorem 3.5. *Assume that η,θ and δ are congruences on a finite algebra* $\mathbf{A}$, $\delta \prec \theta$ *and* $\operatorname{typ}(\delta,\theta) \in \{\mathbf{2},\mathbf{3},\mathbf{4}\}$. *Assume that $U \in \mathrm{M}_{\mathbf{A}}(\delta,\theta)$ has body B and tail T. The following conditions are equivalent*:

(i) $C(\eta,\theta;\delta)$,

(ii) $C(\theta,\eta;\delta)$ *and* $\eta|_U \subseteq B^2 \cup T^2$,

(iii) $C(\theta,\eta;\delta)$ *and* $\eta \overset{s}{\sim} \eta \wedge (\delta:\theta)$.

(This theorem is a combination of lemmas and remarks from [6].)

Now, in the argument which we used to prove the finitely generated version of Jónsson's Lemma and Lemma 3.2 we are guaranteed by Lemma 3.4 that in **Con B** it is the case that $C(\theta,\eta_i;\delta)$ holds for some i. Furthermore, from $C(\eta_i,\theta;\delta)$ one can finish the proof of each lemma. To prove our Jónsson's-type lemma, let us analyze situations where $C(\theta,\eta;\delta)$ holds for some η while $C(\eta,\theta;\delta)$ fails.

The assumption that $C(\eta,\theta;\delta)$ fails is equivalent to $\eta \not\leq (\delta:\theta)$. Hence, there are $\alpha,\beta \in \text{Con }\mathbf{B}$ such that $\alpha \prec \beta \leq \eta$, $\alpha \leq (\delta:\theta)$ and $\beta \not\leq (\delta:\theta)$. (Any such pair will do, but a specific α which works is $\alpha = \eta \wedge (\delta:\theta)$ and for this α we may take β to be any congruence for which $\alpha \prec \beta \leq \eta$.) Fig. 1 illustrates the order relationship between all the congruences of **Con B** mentioned so far. (Fig. 1 is plausible when $\operatorname{typ}(\delta,\theta) = \mathbf{2}$, but when $\operatorname{typ}(\delta,\theta) \in \{\mathbf{3},\mathbf{4}\}$ we must lower $(\delta:\theta)$, α and β so that $(\delta:\theta) = \delta$.) Since $\beta \leq \eta$ we get that $C(\theta,\beta;\delta)$ holds. We cannot have $C(\beta,\theta;\delta)$, since $\beta \not\leq (\delta:\theta)$. By Theorem 3.5 we find that $\beta \overset{s}{\not\sim} \beta \wedge (\delta:\theta) = \alpha$. Hence, $\operatorname{typ}(\alpha,\beta) \in \{\mathbf{3},\mathbf{4},\mathbf{5}\}$. Note that $\langle\alpha,\beta\rangle$ is perspective with some prime quotient in the interval $I[(\delta:\theta),1]$, hence with some prime quotient $\langle\sigma,\rho\rangle$ in $I[\delta,1]$. But since $\mathbf{B}/\delta \cong \mathbf{A}$, we get that

$$\operatorname{typ}(\alpha,\beta) = \operatorname{typ}(\sigma,\rho) \in \operatorname{typ}\{\delta,1\} \subseteq \operatorname{typ}\{\mathbf{A}\} \subseteq \{\mathbf{1},\mathbf{2},\mathbf{3},\mathbf{4}\}.$$

It follows that $\operatorname{typ}(\alpha,\beta) \in \{\mathbf{3},\mathbf{4}\}$. We summarize what we know about **Con B** so far (assuming the hypotheses of Lemma 3.3 and that $C(\theta,\eta;\delta)$ holds while $C(\eta,\theta;\delta)$ fails):

(i) $\operatorname{typ}(\delta,\theta) \in \{\mathbf{2},\mathbf{3},\mathbf{4}\}$,

(ii) $\operatorname{typ}(\alpha,\beta) \in \{\mathbf{3},\mathbf{4}\}$ and

(iii) $C(\theta,\eta;\delta)$.

In the rest of this section we shall prove that these three conditions permit the construction of a proper class of subdirectly irreducible algebras in $\mathsf{V}(\mathbf{B}) \subseteq \mathsf{V}(K)$. As this is contrary to our hypothesis in Lemma 3.3, we shall be able to conclude that with the hypotheses of Lemma 3.3 we have

$$C(\theta,\eta;\delta) \Rightarrow C(\eta,\theta;\delta).$$

We can then finish the proof of Lemma 3.3 in the same way that we finished the proofs of Lemmas 3.1 and 3.2.

Let $\{0,1\}$ be an $\langle\alpha,\beta\rangle$-trace. Since $\operatorname{typ}(\alpha,\beta) \in \{\mathbf{3},\mathbf{4}\}$ we have that $\mathbf{B}|_{\{0,1\}}$ is a minimal algebra of type **3** or **4**. Furthermore, since $(0,1) \in \eta$, we get that $C(\theta,\{0,1\}^2;\delta)$ holds. Finally,

$$(0,1) \in \beta - \alpha = \beta - (\delta : \theta),$$

so $(0,1) \notin (\delta : \theta)$ and in particular $(0,1) \notin \delta$ $(\subseteq (\delta : \theta))$.

The next two theorems indicate why this situation is impossible in a residually small variety. We maintain the notation of our discussion above.

Theorem 3.6. *Let* **B** *be a finite algebra which has congruences* $\delta \prec \theta$ *with* δ *a meet-irreducible congruence. Assume that* **B** *has a pair of elements* $(0,1) \notin \delta$ *and that the following conditions hold:*

(i) $\mathbf{B}|_{\{0,1\}}$ *is a minimal algebra of type* **3** *or* **4**,

(ii) $\operatorname{typ}(\delta,\theta) = \mathbf{2}$ *and*

(iii) $C(\theta,\{0,1\}^2;\delta)$.

Then $\mathsf{V}(\mathbf{B})$ *is residually large.*

Proof. We shall only prove the case of the theorem where $\delta = 0$. For if we factor by δ, the hypotheses remain unaffected and our proof will apply in this case. This will prove that $\mathsf{V}(\mathbf{B}/\delta)$ is residually large, and therefore that $\mathsf{V}(\mathbf{B})$ is residually large.

$\mathbf{B}/\delta \cong \mathbf{A}$, according to our established notation, and θ/δ corresponds to μ; so we need to prove the following. If

(i) $\mathbf{A}|_{\{0,1\}}$ is a minimal algebra of type **3** or **4**,

(ii) $\operatorname{typ}(0,\mu) = \mathbf{2}$ and

(iii) $C(\mu,\{0,1\}^2;0)$;

then $\mathsf{V}(\mathbf{A})$ is residually large.

Note that $(0,1) \notin \mu$ since $\langle 0,1\rangle$ is a 2-snag and $\operatorname{typ}(0,\mu) = \mathbf{2}$. Hence $\operatorname{Cg}(0,1) > \mu$. We are now in precisely the situation of Lemma 10.2 of [3]. In first two paragraphs of Lemma 10.2 of [3], Hobby and McKenzie reduce the hypotheses of their lemma to three statements. Those statements are precisely the three conditions we have enumerated in the last paragraph along with $\mathrm{Cg}(0,1) > \mu$. Their proof shows how to construct a proper class of subdirectly irreducible algebras in $\mathsf{V}(\mathbf{A})$, so our work has been done for us. $\square$

Theorem 3.7. *Let* $\mathbf{B}$ *be a finite algebra which has congruences* $\delta \prec \theta$ *with* δ *a meet-irreducible congruence. Assume that* $\mathbf{B}$ *has a pair of elements* $(0,1) \notin \delta$ *and that the following conditions hold:*

(i) $\mathbf{B}|_{\{0,1\}}$ *is a minimal algebra of type* $\mathbf{3}$ *or* $\mathbf{4}$,

(ii) $\mathrm{typ}(\delta,\theta) \in \{\mathbf{3},\mathbf{4}\}$ *and*

(iii) $C(\theta,\{0,1\}^2;\delta)$.

Then $\mathsf{V}(\mathbf{B})$ *is residually large.*

Proof. As in the last theorem, it suffices to work in $\mathbf{A}$. Therefore, we replace $\mathbf{B}$ by $\mathbf{A}$, δ by 0 and θ by μ. Choose $U \in \mathrm{M}_{\mathbf{A}}(0,\mu)$ and let B and T be the body and tail of U, respectively. Choose $e \in E(\mathbf{A})$ such that $e(A) = U$. To further set notation for this proof, let $B = \{u,z\}$ and let $x \vee y$ and $x \wedge y$ denote the pseudo-join and pseudo-meet polynomials of $\mathbf{A}|_U$ with respect to the ordering $z < u$. Let $q, p \in \mathrm{Pol}_2\mathbf{A}$ be lattice polynomials on $\{0,1\}$. Say, $q(0,0) = q(1,0) = q(0,1) = 0 = p(0,0)$ and $q(1,1) = 1 = p(0,1) = p(1,0) = p(1,1)$.

Since $(u,z) \in \mu \leq \mathrm{Cg}(0,1)$, we get $\mu|_U \leq \mathrm{Cg}(0,1)|_U$ and so there is a polynomial $f \in \mathrm{Pol}_1\mathbf{A}$ such that $ef(0) = u \neq ef(1)$ or else $ef(1) = u \neq ef(0)$. Both arguments are symmetric, so we assume that $ef(0) = u \neq ef(1)$. (Incidentally, the symmetry of these two arguments follows from the fact that $\mathbf{A}|_{\{0,1\}}$ has both meet and join polynomials. It would not be enough in our argument for $\mathbf{A}|_{\{0,1\}}$ to have a binary semilattice polynomial.) The fact that $C(\mu,\{0,1\}^2;0)$ holds implies that $C(\{u,z\}^2,\{ef(0),ef(1)\}^2;0)$ holds and $C(\{u,z\}^2,\{u,z\}^2;0)$ does not hold. Hence $ef(1) \in U - B = T$. Let $v = ef(1)$. Let $w = z \wedge v$. Note that

$$w = z \wedge v \;\mu\; u \wedge v = v.$$

Since $v \in T$ we have $(v,w) \in \mu|_T = 0_T$, that is $v = w$.

Now we begin a construction which shows that $\mathsf{V}(\mathbf{A})$ is residually large, contrary to our hypothesis. We define certain elements of $\mathbf{A}^\kappa$: 0^i is the element $(c_j)_{j<\kappa} \in A^\kappa$ where $c_j = 0$ for all $j \neq i$ and $c_i = 1$. u^i is the element $(c_j)_{j<\kappa} \in A^\kappa$ where $c_j = u$ for all $j \neq i$ and $c_i = v$. z^i is the element $(c_j)_{j<\kappa} \in A^\kappa$ where $c_j = z$ for all $j \neq i$ and $c_i = v$. If $x \in A$, we write $\hat{x}$ to denote the element $(c_j)_{j<\kappa} \in A^\kappa$ with $c_j = x$ for all j. If $g(\bar{x}) \in \mathrm{Pol}\,\mathbf{A}$, and $g(\bar{x}) = t^{\mathbf{A}}(\bar{x},a_0,\ldots,a_n)$ for some term t and some $a_i \in A$, then we will write $\hat{g}(\bar{x})$ to denote the polynomial of $\mathbf{A}^\kappa$ which is equal to $t^{\mathbf{A}^\kappa}(\bar{x},\hat{a_0},\ldots,\hat{a_n})$. Let $\mathbf{C}$ be the subalgebra of $\mathbf{A}^\kappa$ generated by all elements of the form $\hat{x}$, $x \in A$, and all elements of the form 0^i, $i < \kappa$. The universe of $\mathbf{C}$ contains all elements of the form u^i $(= \hat{e}\hat{f}(0^i))$ and z^i $(= \hat{z}\hat{\wedge}u^i)$. Let $a = \hat{u}$, $b = \hat{z}$, $X = \{0^i \mid i < \kappa\}$ and define

$$\gamma = \mathrm{Cg}^{\mathbf{C}}(\{(u^i,z^i) \mid i < \kappa\}).$$

We claim that (a,b,X,γ) is a 4-tuple which witnesses the fact that $\mathbf{C}$ has a subdirectly irreducible homomorphic image of cardinality $\geq \kappa$.

In this paragraph we show that if $\psi \geq \gamma$ and $|X/(\psi|_X)| < \kappa$, then $(a,b) \in \psi$. If $|X/(\psi|_X)| < \kappa$, then we must have $(0^i,0^j) \in \psi$ for some $i \neq j$. Then

$$0^i = \hat{q}(0^i,0^i)\ \psi\ \hat{q}(0^i,0^j) = \hat{0}.$$

Thus,

$$(u^i,\hat{u}) = (\hat{e}\hat{f}(0^i),\hat{e}\hat{f}(\hat{0})) \in \psi\,.$$

This implies that

$$(z^i,\hat{z}) = (\hat{z}\hat{\wedge}u^i,\hat{z}\hat{\wedge}\hat{u}) \in \psi\,.$$

Finally, we get that

$$a = \hat{u}\ \psi\ u^i\ \gamma\ z^i\ \psi\ \hat{z} = b.$$

To finish the proof we must show that $(a,b) \notin \gamma$. Assume instead that $(a,b) = (\hat{u},\hat{z}) \in \gamma$. Then there is a Mal'cev chain $\hat{u} = x_0,\dots,x_k = \hat{z}$. We may apply $\hat{e}$ to every element of this chain and obtain another such chain, so assume that each x_i is a member U^κ. We may of course assume that $x_0 \neq x_1$. Let us show that this leads to a contradiction. Since $\{x_0,x_1\} = \{r(u^i),r(z^i)\}$ for some $r \in \mathrm{Pol}_1\mathbf{C}$ satisfying $r(C) \subseteq U^\kappa$ and $x_0 = \hat{u}$, it will suffice to prove that

$$r(u^i) = \hat{u} \Leftrightarrow r(z^i) = \hat{u}.$$

Both directions of this bi-implication can be proved with the same arguments, so assume that $r(z^i) = \hat{u}$. For some $s(x,\bar{y}) \in \mathrm{Pol}_{m+1}\mathbf{A}$ we may write $r(x) = \hat{s}(x,0^{i_0},\dots,0^{i_{m-1}})$. Choose any $j,k < \kappa$. We have $r(z^i) = \hat{u}$, so

$$(r(z^i))_j = s((z^i)_j,\bar{g}) = u = s((z^i)_k,\bar{h}) = (r(z^i))_k,$$

where $\bar{g},\bar{h} \in \{0,1\}^m$. But $z^i = \hat{z}\hat{\wedge}u^i = \hat{z}\hat{\wedge}\hat{e}\hat{f}(0^i)$. Hence, we can rewrite the last displayed equation as

$$s(z \wedge ef((0^i)_j),\bar{g}) = u = s(z \wedge ef((0^i)_k),\bar{h}).$$

Since $C(\mu,\{0,1\}^2;0)$ holds, we get that

$$s(u \wedge ef((0^i)_j),\bar{g}) = s(u \wedge ef((0^i)_k),\bar{h}).$$

This holds for all $j,k < \kappa$. Working backwards now and using $\hat{u}\hat{\wedge}\hat{e}\hat{f}(0^i) = \hat{u}\hat{\wedge}u^i = u^i$, we get that

$$(r(u^i))_j = (r(u^i))_k$$

for all $j,k < \kappa$. Hence, $r(u^i) = \hat{d}$ for some $d \in U$. That is, $(r(z^i),r(u^i)) = (\hat{u},\hat{d})$. In the ith coordinate this says that (for some $\bar{o} \in \{0,1\}^m$)

$$u = (r(z^i))_i = s(v,\bar{o}) = (r(u^i))_i = d.$$

This shows that $r(z^i) = \hat{u} = \hat{d} = r(u^i)$ as we claimed. Our conclusion is that $(a,b) = (\hat{u},\hat{z}) \notin \gamma$. It follows that $\mathsf{V}(\mathbf{A})$ is residually large. Since $\mathsf{V}(\mathbf{A}) \subseteq \mathsf{V}(\mathbf{B})$ we are done. $\square$

Proof of Lemma 3.3. In our remarks following Theorem 3.5 we assume that K is a finite set of finite algebras and $\mathbf{A} \in \mathsf{V}(K)$ is a finite subdirectly irreducible algebra with monolith μ. We showed that if $\mathbf{A}/(0 : \mu) \notin \mathsf{HS}(K)$, then there exist a non-abelian prime quotient $\langle \alpha, \beta \rangle$ as depicted in Fig. 1. If $\mathrm{typ}(\alpha, \beta) = \mathbf{5}$, we argued that $\mathbf{5} \in \mathrm{typ}\{\mathbf{A}\}$. If $\mathrm{typ}(\alpha, \beta) \in \{\mathbf{3}, \mathbf{4}\}$, then Theorems 3.6 and 3.7 prove that $\mathsf{V}(\mathbf{A})$ is residually large. This concludes the proof. $\square$

4. A Property of $(0 : \mu)$

In this section we prove that if $\mathbf{A}$ is a finite subdirectly irreducible algebra contained in a residually small variety and μ is the monolith of $\mathbf{A}$ where $\mathrm{typ}(0, \mu) = \mathbf{2}$, then $(0 : \mu)$ is abelian.

Lemma 4.1. *Assume that* $\mathbf{A}$ *is finite, that* $\mathbf{A}$ *has a prime quotient* $\langle \delta, \theta \rangle$ *of type* $\mathbf{2}$ *and that* $U \in \mathrm{M}_{\mathbf{A}}(\delta, \theta)$ *has body* B *and tail* T. *Then* $(\delta : \theta)|_U \subseteq B^2 \cup T^2$. *Further,* B *is a single* $(\delta : \theta)|_U$*-class.*

Proof. The first statement follows from Theorem 3.5(i) $\Rightarrow$ (ii) with $\eta = (\delta : \theta)$. For the second statement, the argument of Lemma 4.2 of [6] proves that $(\delta : \theta)$ is the largest congruence γ on $\mathbf{A}$ such that $C(\gamma, \theta|_U; \delta)$ holds. Since $\mathbf{A}|_B$ is nilpotent and $\delta|_U \prec \theta|_U$ we get that $C(\mathrm{Cg}(B^2), \theta|_U; \delta)$ holds, so $B^2 \subseteq (\delta : \theta)|_U$. $\square$

One consequence of Lemma 4.1 is that both $C((\delta : \theta), \theta; \delta)$ and $C(\theta, (\delta : \theta); \delta)$ hold when $\mathrm{typ}(\delta, \theta) = \mathbf{2}$. The first follows from the definition of $(\delta : \theta)$ while the second follows from $(\delta : \theta)|_U \subseteq B^2 \cup T^2$ and Theorem 3.5.

Theorem 4.2. *Assume that* $\mathbf{A}$ *is a finite algebra with congruences* $\delta \prec \theta$ *where* δ *is meet-irreducible and* $\mathrm{typ}(\delta, \theta) = \mathbf{2}$. *Assume also that* $\mathsf{V}(\mathbf{A})$ *is residually small. For any* $\alpha, \beta \in \mathrm{Con}\ \mathbf{A}$ *we have*

$$(C(\alpha, \theta; \delta)\ \&\ C(\theta, \beta; \delta)) \Rightarrow C(\alpha, \beta; \delta).$$

Proof. $C(\alpha, \psi; \delta) \Leftrightarrow C(\alpha \vee \delta, \psi; \delta)$ for any ψ, so we lose no generality by assuming that $\alpha \geq \delta$. Now each of the congruences in question lie above $\beta \wedge \delta$, so factoring by this congruence we may assume that $\beta \wedge \delta = 0$. Let us assume that $C(\alpha, \theta; \delta)$ and $C(\theta, \beta; \delta)$ hold, but that $C(\alpha, \beta; \delta)$ fails. Then $C(\alpha, \beta; 0)$ fails, too, since for any three congruences it is true that

$$C(\alpha, \beta; \beta \wedge \delta) \Rightarrow C(\alpha, \beta; \delta).$$

Let $[\alpha,\beta]$ denote the least congruence χ such that $C(\alpha,\beta;\chi)$. From what we have said and the properties of the centralizer relation, $0 < [\alpha,\beta] \leq \alpha \wedge \beta$. We proceed to argue that $\mathsf{V}(\mathbf{A})$ is residually large.

Since $C(\alpha,\beta;0)$ fails, for some $p \in \mathrm{Pol}_{n+1}\mathbf{A}$ and some pairs $(0,1) \in \alpha$ and $(r_i,s_i) \in \beta$ we have

$$p(0,\bar{r}) = p(0,\bar{s})$$

while

$$g = p(1,\bar{r}) \; [\alpha,\beta] - 0_{\mathbf{A}} \; p(1,\bar{s}) = h.$$

Choose a minimal set $U \in M_{\mathbf{A}}(\delta,\theta)$, a trace $N \subseteq U$ and a pair $(u,v) \in N^2 - \delta$. Since $(g,h) \in [\alpha,\beta] \leq \beta$ and $g \neq h$, we cannot have $(g,h) \in \delta$. As δ is meet-irreducible, this implies that $(u,v) \in \mathrm{Cg}(g,h) \vee \delta$. There is a Mal'cev chain $u = x_0,\ldots,x_n = v$ where for each $i < n$ we have $\{x_i,x_{i+1}\} = \{p_i(g),p_i(h)\}$ or $(x_i,x_{i+1}) \in \delta$. Pick $e \in E(\mathbf{A})$ so that $e(A) = U$. If we apply e to the chain $x_0,\ldots,x_n$, we get another such chain contained in U. In fact, the chain is contained completely inside the body of U since

$$\begin{aligned}(\mathrm{Cg}(g,h) \vee \delta)|_U &= \mathrm{Cg}(g,h)|_U \vee \delta|_U\\ &\leq [\alpha,\beta]|_U \vee \delta|_U\\ &\leq \alpha|_U \vee \delta|_U\\ &\leq (\delta : \theta)|_U\\ &\subseteq B^2 \cup T^2,\end{aligned}$$

where B is the body of U and T is the tail. $(u,v) \notin \delta$ by choice, so there is an i such that $ep_i(g) \neq ep_i(h)$ and both elements belong to the body of U. If we apply ep_i to both of the two displayed equations above which witness a failure of $C(\alpha,\beta;0)$, then we see that no generality is lost in assuming that $p(A,A^n) \subseteq U$ and that all four elements in the previous displayed equations belong to B. We make this assumption.

Let $d(x,y,z)$ be a pseudo-Mal'cev polynomial of U. We assume that $d(x,y,z) = ed(x,y,z)$ so that the range of d is contained in U. Define $p'(x,\bar{y}) = d(p(x,\bar{y}),p(x,\bar{s}), p(1,\bar{s}))$. Using the previous displayed equations and the fact that d is Mal'cev on B we find that

$$p'(0,\bar{s}) = p(1,\bar{s}) = p'(1,\bar{s})$$

and

$$p'(0,\bar{r}) = p(1,\bar{s}) \neq p(1,\bar{r}) = p'(1,\bar{r}).$$

Let us set $l = p'(0,\bar{r}) = p'(0,\bar{s}) = p'(1,\bar{s})$ and $m = p'(1,\bar{r})$. Both l and m belong to B and $(l,m) \in [\alpha,\beta] - 0_{\mathbf{A}}$. Hence, $(l,m) \notin \delta$ just as we argued for the pair (g,h). From this and the fact that $\mathbf{A}|_B$ is Mal'cev, we get that

$$(u,v) \in \theta|_B \leq \mathrm{Cg}(l,m)|_B \circ \delta|_B.$$

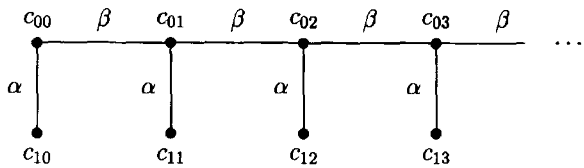

Fig. 2. The "comb" $(c_{0j}, c_{1j})_{j<k}$.

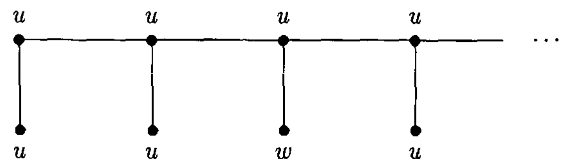

Fig. 3. The comb u^2.

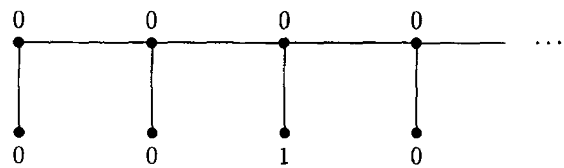

Fig. 4. The comb 0^2.

Hence, there is a $w \in B$ such that $(u, w) \in \mathrm{Cg}(l, m)|_B$ and $(v, w) \in \delta|_B$. Since $\mathbf{A}|_B$ is Mal'cev and $(u, w) \in \mathrm{Cg}(l, m)|_B$ there is a polynomial $f \in \mathrm{Pol}_1\mathbf{A}|_B$ such that $f(l) = u$ and $f(m) = w$. Let $q(x, \bar{y}) = fp'(x, \bar{y})$. Finally, we have

$$q(0, \bar{r}) = q(0, \bar{s}) = q(1, \bar{s}) = u$$

and

$$q(1, \bar{r}) = w.$$

This prepares us to construct algebras in $\mathsf{V}(\mathbf{A})$ having 4-tuples (a, b, X, γ) witnessing the fact that $\mathsf{V}(\mathbf{A})$ is residually $\geq \kappa$ for any cardinal κ.

Let $\mathbf{C}$ be the subalgebra of $\mathbf{A}^\kappa \times \mathbf{A}^\kappa$ whose universe consists of all tuples $(c_{0j}, c_{1j})_{j<\kappa}$ with the properties that

(i) there is a $c \in A$ such that $c_{ij} = c$ for and all but finitely many pairs (i, j), $i =$ 0 or 1 and $j < \kappa$, and

(ii) $c_{00} \mathrel{\beta} c_{0j} \mathrel{\alpha} c_{1j}$ for all $j < \kappa$.

Pictorially, $\mathbf{C}$ is the subalgebra of all "almost constant combs" (see Fig. 2) in $\mathbf{A}^\kappa \times \mathbf{A}^\kappa$.

We will use the notation u^k, $k < \kappa$, to denote the element $(c_{0j}, c_{1j})_{j<\kappa} \in C$ where $c_{ij} = u$ whenever $(i, j) \neq (1, k)$ while $c_{1k} = w$. We will use the notation 0^k to denote the element $(c_{0j}, c_{1j})_{j<\kappa} \in C$ where $c_{ij} = 0$ whenever $(i, j) \neq (1, k)$ while $c_{1k} = 1$. For example, u^2 is the comb pictured in Fig. 3 and 0^2 is the comb pictured in Fig. 4.

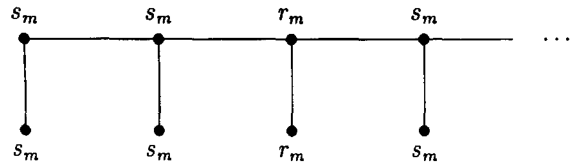

Fig. 5. The comb s_m^2.

For $m < n$ we will use the notation s_m^k to denote the element $(c_{0j}, c_{1j})_{j<\kappa} \in C$ where $c_{ij} = s_m$ whenever $j \neq k$ while $c_{0k} = c_{1k} = r_m$. Notice that there is a difference in the $0k$-coordinate from the way we defined u^k and 0^k. As an example, s_m^2 is the comb in Fig. 5.

Since $(u,w) \in \theta \leq \alpha$, $(0,1) \in \alpha$, $(r_i,s_i) \in \beta$, all elements of the form u^i, 0^i and s_m^i belong to $\mathbf{C}$. We use the notation $\hat{u}$ to denote the comb $(c_{0j}, c_{1j})_{j<\kappa}$ where $c_{ij} = u$ for all i and j. Let $a = \hat{u}$ and let $b = u^0$. Let $X = \{0^i | i < \kappa\}$. Let $\gamma = \mathrm{Cg}^{\mathbf{C}}(\{(u^i,u^j) \in C^2 | i,j < \kappa\})$. We now argue that (a,b,X,γ) is a 4-tuple which witnesses the fact that $\mathbf{C}$ has subdirectly irreducible homomorphic image of cardinality $\geq \kappa$.

In this paragraph we establish that if $\psi \geq \gamma$ and $|X/(\psi|_X)| < \kappa$, then $(a,b) \in \psi$. Assume that $\psi \geq \gamma$ and that $|X/(\psi|_X)| < \kappa$. Since

$$|X| = |\{0^i | i < \kappa\}| = \kappa,$$

it must be that $(0^i, 0^j) \in \psi$ for some $i \neq j$. Let $\bar{s}^j = (s_0^j, s_1^j, \ldots, s_{n-1}^j)$. Then the equalities established for q above guarantee that

$$a = \hat{u} = \hat{q}(0^i, \bar{s}^j)\ \psi\ \hat{q}(0^j, \bar{s}^j) = u^j\ \gamma\ u^0 = b.$$

Thus, $(a,b) \in \psi$.

It remains to show that $(a,b) \notin \gamma$. If this were not so, then we could find a Mal'cev chain

$$a = \hat{u} = x_0, \ldots, x_{m-1} = u^0 = b,$$

where for each i we have $(x_i, x_{i+1}) = (p_i(u^j), p_i(u^k))$ with $p_i \in \mathrm{Pol}_1\mathbf{C}$, $j,k < \kappa$. If we apply $\hat{e}$ to the elements of this chain, we obtain another chain where all elements are among the elements of $N^\kappa \times N^\kappa$. (Recall that $e \in E(\mathbf{A})$ was chosen above so that $e(A) = U$. Since $\gamma \leq \theta^{2\kappa}$, all elements of the chain are $\theta^{2\kappa}$-related to $\hat{u}$ and they belong to U^κ.) Let $+, -, u$ be abelian group polynomials of $\mathbf{A}|_N$. We shall show by induction that, for each $i < m$, if $x_i = (c_{0j}, c_{1j})_{j<\kappa}$ in the previously displayed Mal'cev chain, then

$$\left(\sum_{j<\kappa} c_{1j} \right) \delta\ u.$$

Here the sum is taken in N. (An inductive argument shows that for any x_i all but finitely many of the c_{1j} are equal to u, so the sum of all c_{1j} is at least defined.) The case when $i = 0$ is trivial since $x_0 = \hat{u}$. Hence, we will finish our inductive proof by showing that if $x_i = (c_{0j}, c_{1j})_{j<\kappa}$ and $x_{i+1} = (c'_{0j}, c'_{1j})_{j<\kappa}$, then

$$\left(\sum_{j<\kappa} c_{1j}\right) \delta \left(\sum_{j<\kappa} c'_{1j}\right).$$

Since $(x_i, x_{i+1}) = (p_i(u^j), p_i(u^k))$ and $p_i(x) = \hat{t}(x, \bar{a})$ for some polynomial $t \in \mathrm{Pol}_{l+1}\mathbf{A}$ satisfying $t(A, A^l) \subseteq U$ and some tuple $\bar{a} \in C^l$, we can write

$$x_i = \hat{t}(u^j, a_1, \ldots, a_l)$$

and

$$x_{i+1} = \hat{t}(u^k, a_1, \ldots, a_l).$$

But u^j and u^k are equal at all coordinates other than the $1j$th and $1k$th. The same is true therefore of x_i and x_{i+1}. Thus, it suffices to show that

$$(x_i)_{1j} + (x_i)_{1k} \ \delta \ (x_{i+1})_{1j} + (x_{i+1})_{1k}$$

or more specifically that

$$d((x_i)_{1j}, (x_{i+1})_{1j}, (x_i)_{1k}) \ \delta \ (x_{i+1})_{1k}.$$

We are using the fact that $d(x, y, z) = x - y + z$ for elements $x, y, z \in N$. Assume instead that

$$d((x_i)_{1j}, (x_{i+1})_{1j}, (x_i)_{1k}) \ \not\delta \ (x_{i+1})_{1k}.$$

Written in another way, this is

$$d(t(w, \bar{a}_{1j}), t(u, \bar{a}_{1j}), t(u, \bar{a}_{1k})) \ \not\delta \ t(w, \bar{a}_{1k}) = d(t(w, \bar{a}_{1j}), t(w, \bar{a}_{1j}), t(w, \bar{a}_{1k})).$$

Changing all occurrences of $\bar{a}_{1j}$ to $\bar{a}_{0j}$ and $\bar{a}_{1k}$ to $\bar{a}_{0k}$ and using the facts that $C(\alpha, \theta; \delta)$ and that for each i we have $((a_i)_{0j}, (a_i)_{1j})$, $((a_i)_{0k}, (a_i)_{1k}) \in \alpha$ we get that

$$d(t(w, \bar{a}_{0j}), t(u, \bar{a}_{0j}), t(u, \bar{a}_{0k})) \ \not\delta \ d(t(w, \bar{a}_{0j}), t(w, \bar{a}_{0j}), t(w, \bar{a}_{0k})).$$

Observe that since, say, $t(w, \bar{a}_{1j}) \in B$, $t(w, \bar{a}_{0j}) \in U$ and $((a_i)_{0j}, (a_i)_{1j}) \in \alpha$, we even have $t(w, \bar{a}_{0j}) \in B$. Here we are using that $\alpha|_U \leq (\delta : \theta)|_U \subseteq B^2 \cup T^2$. This argument shows that $t(w, \bar{a}_{0j})$, $t(w, \bar{a}_{0k})$, $t(u, \bar{a}_{0j})$ and $t(u, \bar{a}_{0k})$ all belong to B. Since d is Mal'cev on B this leads to

$$d(t(w, \bar{a}_{0j}), t(u, \bar{a}_{0j}), t(u, \bar{a}_{0j})) = d(t(w, \bar{a}_{0j}), t(w, \bar{a}_{0j}), t(w, \bar{a}_{0j})).$$

Define $z(x, \bar{y}) = d(t(w, \bar{a}_{0j}), t(x, \bar{a}_{0j}), t(x, \bar{y}))$. Here is a summary of our knowledge of z:

$$z(u, \bar{a}_{0j}) = z(w, \bar{a}_{0j})$$

while

$$z(u,\bar{a}_{0k}) \not\delta\, z(w,\bar{a}_{0k})$$

and all four elements belong to B. Define $z'(x,\bar{y}) = d(z(x,\bar{y}),z(w,\bar{y}),z(w,\bar{a}_{0k}))$. We compute that

$$z'(w,\bar{a}_{0j}) = z(w,\bar{a}_{0k}) = z'(w,\bar{a}_{0k})$$

while

$$z'(u,\bar{a}_{0j}) = z(w,\bar{a}_{0k}) \not\delta\, z(u,\bar{a}_{0k}) = z'(u,\bar{a}_{0k}).$$

But for each $i < l$ we have $((a_i)_{0j},(a_i)_{0k}) \in \beta$, so this is a failure of $C(\theta,\beta;\delta)$. This contradiction invalidates our assumption that

$$d((x_i)_{1j},(x_{i+1})_{1j},(x_i)_{1k}) \not\delta\, (x_{i+1})_{1k}.$$

We conclude that

$$(x_i)_{1j} + (x_i)_{1k}\ \delta\ (x_{i+1})_{1j} + (x_{i+1})_{1k}.$$

By induction we find that for any $i < m$, if $x_i = (c_{0j},c_{1j})_{j<\kappa}$, then

$$\left(\sum_{j<\kappa} c_{1j}\right) \delta\ u.$$

In particular, this must hold for $x_{m-1} = u^0$. But $u^0 = (c_{0j},c_{1j})_{j<\kappa}$ where all $c_{ij} = u$ except $c_{10} = w$. It follows that for $x_{m-1} = u^0$ we have

$$\sum_{j<\kappa} c_{1j} = w\ \delta\ v \not\delta\ u.$$

This is a contradiction to our assumption that $(a,b) \in \gamma$. In other words, $(a,b) \notin \gamma$ as we claimed and the proof is finished. □

Corollary 4.3. *Assume that* $\mathbf{A}$ *is a finite subdirectly irreducible algebra with monolith* μ *and that* $\mathrm{typ}(0,\mu) = \mathbf{2}$. *If* $(0 : \mu)$ *is nonabelian, then* $\mathsf{V}(\mathbf{A})$ *is residually large.*

Proof. From Lemma 4.1 and the remarks that follow it $C((0 : \mu),\mu;0)$ and $C(\mu,(0 : \mu);0)$ hold. From Theorem 4.2 $\neg C((0 : \mu),(0 : \mu);0)$ implies that $\mathsf{V}(\mathbf{A})$ is residually large. □

5. Cardinality bounds

Theorem 5.1. *Assume that* $\mathbf{A}$ *is a finite subdirectly irreducible algebra with monolith* μ *and that* $\mathrm{typ}(0,\mu) = \mathbf{2}$. *If* α *is an abelian congruence on* $\mathbf{A}$ *of index* n, *then*

$$|A| \leq n \cdot m^m,$$

where $m = |F_{\mathsf{V}(\mathbf{A})}(n+1)|$.

Proof. Let Y be an α-class of maximum cardinality. Since α has index n, it will suffice to show that $|Y| \leq m^m$. We assume that $\alpha > 0$.

Choose $U \in \mathrm{M}_{\mathbf{A}}(0,\mu)$, $e \in E(\mathbf{A})$ such that $e(A) = U$ and a pair of elements $(0,1) \in \mu|_U - 0_U$. Let V denote the $\alpha|_U$-class containing 0. Note that since $(0,1) \in \mu|_U \leq \alpha|_U$ we have $\{0,1\} \subseteq V$. $C(\alpha,\alpha;0)$ holds, so we have $C(\alpha,\mu;0)$ and therefore $\alpha \leq (0:\mu)$. This implies that $\alpha|_U \subseteq B^2 \cup T^2$ where B is the body of U and T is the tail. $0 \in B$, so $V \subseteq B$.

Fix a transversal for α: $\{c_0,\ldots,c_{n-1}\}$. Let F denote the subset of $\mathrm{Pol}_1\mathbf{A}$ consisting of all polynomials of the form $p(x,c_0,\cdots,c_{n-1})$ where $p(x,\bar{y}) \in \mathbf{Clo}_{n+1}\mathbf{A}$. Since $m = |F_{\mathsf{V}(\mathbf{A})}(n+1)|$, we have $|F| \leq m$.

Choose distinct elements $u,v \in Y$. Since $(0,1) \in \mathrm{Cg}^{\mathbf{A}}(u,v)|_U$ and $U = e(A)$, we can find $q(x,\bar{y}) \in \mathbf{Clo}_{\ell+1}\ \mathbf{A}$ and $\bar{a} \in A^{\ell}$ such that $eq^{\mathbf{A}}(u,\bar{a}) = 0 \neq eq^{\mathbf{A}}(v,\bar{a})$ or the same with u and v switched. For each a_i choose $b_i \in \{c_0,\ldots,c_{n-1}\}$ such that $(a_i,b_i) \in \alpha$. Then since $C(\alpha,\alpha;0)$, $(u,v) \in \alpha$ and

$$eq^{\mathbf{A}}(u,\bar{a}) \neq eq^{\mathbf{A}}(v,\bar{a}),$$

we get

$$eq^{\mathbf{A}}(u,\bar{b}) \neq eq^{\mathbf{A}}(v,\bar{b}).$$

Each b_i is a member of $\{c_0,\ldots,c_{n-1}\}$ so $q^{\mathbf{A}}(x,\bar{b})$ equals some $q'(x) \in F$. Now we have

$$eq'(v)\ \alpha|_U - 0_{\mathbf{A}}\ eq'(u) = eq^{\mathbf{A}}(u,\bar{b})\ \alpha|_U\ eq(u,\bar{a}) = 0 \in V,$$

so $eq'(v)$ and $eq'(u)$ are distinct members of V. We also have that $eq'(Y) \subseteq V$ since Y is an α-class and V is an $\alpha|_U$-class.

Let F' be the subset of F consisting of polynomials $p(x) \in F$ such that $ep(Y) \subseteq V$. Define a function $\Phi : Y \to V^{|F'|}$ as follows:

$$\Phi(w) = (ep(w))_{p \in F'}\,.$$

In the last paragraph we showed that for any $u \neq v$ in Y there is a $q'(x) \in F'$ such that $eq'(u) \neq eq'(v)$. It follows that the function Φ described in the last displayed equation is 1–1. Hence $|Y| \leq |V|^{|F'|} \leq |V|^m$. It remains to show that $|V| \leq m$.

$V \subseteq B$, so $\mathbf{A}|_V$ is Mal'cev. Since α is abelian, $\alpha|_V = 1_V$ is abelian. We get that $\mathbf{A}|_V$ is affine, since any abelian Mal'cev algebra is affine. The algebra $\mathbf{A}|_V$ has a least nonzero congruence since (by Lemma 2.4 of [3]) restriction of congruences is a homomorphism from the interval $I[0,\alpha]$ in $\mathbf{Con\ A}$ onto $\mathbf{Con\ A}|_V$ and $\mu|_V > 0_V$. This shows that $\mathbf{A}|_V$ is polynomially equivalent to a subdirectly irreducible module over a finite ring, $\mathbf{R}$. As shown in [5], this implies that $|V| \leq |R|$. The elements of $\mathbf{R}$ may be identified with the unary module polynomials that fix the additive identity element. If we take $0 \in V$ to be the additive identity element of the module structure of $\mathbf{A}|_V$, then we may consider the elements of $\mathbf{R}$ to be the unary polynomials $r(x)$ of $\mathbf{A}|_V$ which satisfy $r(0) = 0$. Suppose that $r(x) = s^{\mathbf{A}}(x,\bar{g})$, $s \in \mathbf{Clo}_{m+1}\mathbf{A}$, $\bar{g} \in A^m$, is such a polynomial. For each g_i we choose $h_i \in \{c_0, \ldots, c_{n-1}\}$ such that $(g_i, h_i) \in \alpha$. If d is the pseudo-Mal'cev polynomial of U, then d is Mal'cev on $V \subseteq B$. We have $es^{\mathbf{A}}(0, \bar{g}) = 0 \in V$, so when $x \in V$

$$d(es^{\mathbf{A}}(0,\bar{g}), es^{\mathbf{A}}(0,\underline{\bar{g}}), es^{\mathbf{A}}(x,\bar{g})) = es^{\mathbf{A}}(x,\bar{g}) = d(es^{\mathbf{A}}(x,\bar{g}), es^{\mathbf{A}}(x,\underline{\bar{g}}), es^{\mathbf{A}}(x,\underline{\bar{g}}))$$

since all the elements in this equation of the form $es^{\mathbf{A}}(-,-)$ belong to $V \subseteq B$. (This observation is based on the fact that all such elements are $\alpha|_U$-related to 0 and $0/\alpha|_U = V$.) From $C(\alpha, \alpha; 0)$, we can change each $\underline{\bar{g}}$ to $\bar{h}$ and get

$$d(es^{\mathbf{A}}(0,\bar{g}), es^{\mathbf{A}}(0,\bar{h}), es^{\mathbf{A}}(x,\bar{h})) = d(es^{\mathbf{A}}(x,\bar{g}), es^{\mathbf{A}}(x,\bar{h}), es^{\mathbf{A}}(x,\bar{h})).$$

The right-hand side equals $es^{\mathbf{A}}(x,\bar{g})$ while the left-hand side equals $d(0, es^{\mathbf{A}}(0,\bar{h}), es^{\mathbf{A}}(x,\bar{h}))$. Since $e|_V = id_V$, we get that for $x \in V$ it is the case that

$$r(x) = es^{\mathbf{A}}(x,\bar{g}) = d(0, es^{\mathbf{A}}(0,\bar{h}), es^{\mathbf{A}}(x,\bar{h})).$$

But $s^{\mathbf{A}}(x,\bar{h}) \in F$ since each $h_i \in \{c_0, \ldots, c_{n-1}\}$. What we have shown in this paragraph is that for every $r(x)$ which represents an element of $\mathbf{R}$ there is an element $w(x) = s^{\mathbf{A}}(x,\bar{h}) \in F$ such that $r(x) = d(0, ew(0), ew(x))$. Hence the function

$$\psi : F \to R : w(x) \mapsto d(0, ew(0), ew(x))$$

is onto. This shows that

$$|V| \le |R| \le |F| \le m.$$

From our earlier arguments we get that $|Y| \le m^m$ and this finishes the proof. □

Theorem 5.2. *Assume that* $\mathbf{A}$ *is an algebra of cardinality* n *and that* $\mathsf{V}(\mathbf{A})$ *is residually small. Let* $\mathbf{B} \in \mathsf{V}(\mathbf{A})$ *be a finite subdirectly irreducible algebra in* $\mathsf{V}(\mathbf{A})$ *with monolith* μ *where* $\operatorname{typ}(0,\mu) \neq \mathbf{1}$ *and* $\mathbf{5} \notin \operatorname{typ}\{\mathbf{B}\}$. *Then*

$$|B| \le n^{n^{n^{n+2}}}.$$

Proof. From $\operatorname{typ}(0,\mu) \in \{\mathbf{2},\mathbf{3},\mathbf{4}\}$ and Theorem 4.2 we get that $(0 : \mu)$ is abelian. By Lemma 3.3, the index of $(0 : \mu)$ is at most $|A| = n$. This already shows that $|B| \le |A| = n$ if $\operatorname{typ}(0,\mu) \in \{\mathbf{3},\mathbf{4}\}$ since $(0 : \mu) = 0$ in this case. (In fact, $\mathbf{B} \in \mathsf{HS}(\mathbf{A})$ in this case.) If $\operatorname{typ}(0,\mu) = \mathbf{2}$, then Theorem 5.1 shows that $|B| \le n \cdot m^m$ where $m = |F_{\mathsf{V}(\mathbf{A})}(n+1)|$. Using the estimate $m \le n^{n^{n+1}}$, which holds in any variety generated by an n-element algebra, one computes that $|B| \le n^{n^{n^{n+2}}}$ as claimed. □

This completes the proof of our main result. Theorem 5.2 describes a recursive function of $|A|$ which bounds the size of certain subdirectly irreducible algebras in $\mathsf{V}(\mathbf{A})$ when this variety is residually small. It is known from McKenzie's recent work in [7] that there does not exist a recursive function which bounds the size of every subdirectly irreducible algebra in $\mathsf{V}(\mathbf{A})$ when this variety is residually small. His construction works for certain algebras with type **5** quotients. This leaves open the following question: Assume that **A** is finite and $\mathsf{V}(\mathbf{A})$ is residually small. Is there a recursive function f such that every subdirectly irreducible in $\mathsf{V}(\mathbf{A})$ which omits type **5** has cardinality $\le f(|A|)$?

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